Gaussian states on quantum groups

Adam Skalski

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General aim

We will discuss some recent (and not so recent) results concerning the abstract notion of Gaussian states on quantum groups, as introduced and studied by Michael Schürmann in 1980s.

(mainly based on joint work with Uwe Franz and Amaury Freslon)

Noncommutative dictionary, quantum probability and quantum states

The idea that quantum spaces should be described in terms of operator algebras, viewed as algebras of functions on quantum spaces, clearly inspired by discoveries of Gelfand, Naimark, Murray, von Neumann and others dating back to 1940s, gained prominence in early 1980s.

unital C^*-algebra A	$\longleftrightarrow$	noncommutative compact space $\mathbb{X}$
state on A	$\longleftrightarrow$	noncommutative probability measure on $\mathbb{X}$

Already this basic setup (possibly letting A be a general unital *-algebra) leads to many quantum probabilistic ideas, especially around the notion of independence.

Noncommutative Gaussian states?

Gaussian measures (on $\mathbb{R}$, or more generally on a locally compact group) form undoubtedly the most important class of probability measures, appearing in a variety of contexts. They have numerous characterisations and properties; in particular they are infinitely divisible, that is they have convolution roots of arbitrary degree, and even embeddable, that is form elements of convolution semigroups of measures.

To consider convolution of quantum measures (i.e. states on a unital *-algebra), we need some more structure.

Bialgebras and convolution

Definition

A *-bialgebra A is a unital *-algebra equipped with a coproduct, i.e. a unital *-homomorphism $\Delta : A \to A \odot A$ which is coassociative:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$

and a counit, i.e. a unital *-homomorphism $\epsilon : A \to \mathbb{C}$ such that

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}.$$

A key toy example: $\mathrm{Pol}(G)$, where G is a compact group: the algebra of all matrix coefficients of finite-dimensional representations of G with

$$\Delta(f)(g,h) = f(g \cdot h), \quad f \in F(G), g, h \in G$$

– identify elements of $\mathrm{Pol}(G) \odot \mathrm{Pol}(G)$ with functions on $G \times G$.
Given a bialgebra A and two states $\omega, \mu \in S(A)$ we define their convolution:

$$\omega \star \mu := (\omega \otimes \mu) \circ \Delta.$$

Compact quantum groups

We will primarily work with a special class of *-bialgebras; these associated with compact quantum groups.

Definition

We call a Hopf*-algebra A a CQG-algebra, and denote it $\mathrm{Pol}(\mathbb{G})$, thinking of $\mathbb{G}$ as a compact quantum group, if A is spanned by coefficients of its finite-dimensional unitary corepresentations.

Examples:

- $\mathrm{Pol}(G)$ for a classical compact group G;
- $\mathbb{C}[\Gamma]$ for a classical discrete group Γ;
- q-deformations of $\mathrm{Pol}(G)$, such as $\mathrm{Pol}(SU_q(2))$;
- liberations of $\mathrm{Pol}(G)$, such as $\mathrm{Pol}(U_N^+)$.

Convolution semigroups of states and their generators

A family $(\mu_t)_{t\geq 0_+}$ of states (positive, unital functionals) on $A = \mathrm{Pol}(\mathbb{G})$ is called a convolution semigroup of states if

1. $\mu_{t+s} = \mu_t \star \mu_s, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in A.$

Such convolution semigroups admit 'pointwise derivatives at 0':

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in A.$$

(note their existence follows from the fundamental theorem of coalgebra)

Quantum Lévy processes

Such convolution semigroups of states are naturally viewed as 'families of distributions of quantum Lévy processes'. This point of view allowed Schürmann to prove his reconstruction theorem.

Theorem (Quantum Schönberg's correspondence)

If $(\mu_t)_{t\geq 0}$ is a convolution semigroup of states on a *-bialgebra A, and

$$\gamma(a) = \lim_{t\to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in A,$$

then $\gamma : A \to \mathbb{C}$ is Hermitian ($\gamma(a^*) = \overline{\gamma(a)}$), vanishes at 1 and is conditionally positive: if $\epsilon(a) = 0$ then $\gamma(a^*a) \geq 0$. Conversely, each such functional on a *-bialgebra is indeed a generating functional of a convolution semigroup of states:

$$\mu_t(a) = \exp_*(t\gamma)(a) := \sum_{n=0}^{\infty} \frac{(t\gamma)^{*n}(a)}{n!}, \quad t \geq 0, a \in A.$$

Schürmann's classification of quantum Lévy processes/generators

A convolution semigroup of states determines the associated quantum Lévy process, and conversely, is determined by its generating functional. This leads to the idea of classifying Lévy processes via their generators, formally applying also to generating functionals on a unital *-algebra with a character.

Definition

A generating functional $\gamma : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is called Gaussian if for any $a, b, c \in \mathrm{Ker}(\epsilon)$ we have $\gamma(abc) = 0$; in other words,

$$\gamma|_{K_3} = 0,$$

where

$$K_n := \mathrm{Lin}\{b_1 \cdots b_n : b_i \in \mathrm{Ker}(\epsilon)\}.$$

Gaussian states

Definition

A generating functional $\gamma : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is called Gaussian if for any $a, b, c \in \mathrm{Ker}(\epsilon)$ we have $\gamma(abc) = 0$; in other words,

$$\gamma|_{K_3} = 0.$$

This is motivated by thinking of quadratic/second order generators (classically we just look at generating functionals which vanish on polynomial functions whose second derivative at the origin is 0).

Definition

A state $\omega \in S(\mathrm{Pol}(\mathbb{G}))$ is called Gaussian, if $\omega = \mu_1$ for some convolution semigroup $(\mu_t)_{t \geq 0}$ of states on $\mathrm{Pol}(\mathbb{G})$ with a Gaussian generating functional.

Gaussian states have many interesting properties: for example they factor through the commutation ideal if and only if they are tracial.

Gaussian states for $SU_q(2)$

Theorem (Skeide)

Let $q \in (-1, 0) \cup (0, 1)$. Then Gaussian generating functionals on $\mathrm{Pol}(SU_q(2))$ can be fully described and in fact all 'live' on $\mathrm{Pol}(\mathbb{T})$ (so come from classical Gaussian processes on $\mathbb{T}$).

Definition

Given two compact quantum groups $\mathbb{H}$, $\mathbb{G}$ we say that $\mathbb{H}$ is a (closed) quantum subgroup of $\mathbb{G}$ if there is a surjective Hopf*-morphism $q : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H})$.

Example: $\mathbb{T}$ is a quantum subgroup of $SU_q(2)$, and every Gaussian generating functional, $\gamma : \mathrm{Pol}(SU_q(2)) \to \mathbb{C}$ factors via $\mathbb{T}$: there is a (Gaussian) generating functional $\tilde{\gamma} : \mathrm{Pol}(\mathbb{T}) \to \mathbb{C}$ such that

$$\gamma = \tilde{\gamma} \circ q.$$

In general quantum subgroups can be identified with Hopf ideals of $\mathrm{Pol}(\mathbb{G})$.

Notion of a Gaussian part

Definition

Let $\mathbb{G}$ be a compact quantum group. We call $\mathbb{H}$ the Gaussian part of $\mathbb{G}$ if $\mathbb{H}$ is the smallest quantum subgroup of $\mathbb{G}$ through which all Gaussian generating functionals (equivalently, all Gaussian states) of $\mathrm{Pol}(\mathbb{G})$ factor.
We write $\mathbb{H} = \text{Gauss}(\mathbb{G})$ and say that $\mathbb{G}$ is Gaussian if $\mathbb{G}$ is its own Gaussian part ($\mathbb{G} = \text{Gauss}(\mathbb{G})$).

So the Gaussian part of $\mathbb{G}$ is morally the subgroup of $\mathbb{G}$ generated by the support of all the Gaussian states of $\mathbb{G}$.

The notion has all the expected functorial properties; for example if $\mathbb{H} \subset \mathbb{G}$, then $\text{Gauss}(\mathbb{H}) \subset \text{Gauss}(\mathbb{G})$, etc...

Classical case and connectedness, take I

Theorem

If G is a classical compact group, then $\text{Gauss}(G)$ coincides with the connected component of G.

This is very easy to show for G being a Lie group, and in general requires some topological considerations.
As expected, Gaussianity is in general related to connectedness.

Proposition

If $\mathbb{G}$ is a finite quantum group, then $\text{Gauss}(\mathbb{G}) = \{e\}$.

One way of showing this is by proving that if a Gaussian generating functional is *bounded*, then it must be trivial.

Kac property

Definition

We say that a compact quantum group is Kac (or of Kac type) if the antipode $S : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ is involutive (equivalently the Haar state of $\mathbb{G}$ is tracial).

Every compact quantum group $\mathbb{G}$ admits the largest Kac quantum subgroup, denoted $\mathrm{Kac}(\mathbb{G})$ (as noted by Sołtan, Tomatsu, Vaes).

Theorem

For any compact quantum group $\mathbb{G}$ we have

$$\mathrm{Gauss}(\mathbb{G}) \subset \mathrm{Kac}(\mathbb{G})$$

(i.e. Gaussian states live on Kac quantum subgroups).

The proof uses a simple, but useful observation: $\mathcal{I}_{Kac}$ is the ideal generated by the range of $S^2 - I$ (or by the range of $\tau_t - I$, where $t \neq 0$ and τ_t is the scaling automorphism).

Kac property and Gaussian parts

Note that already the last two results imply that $\mathrm{Gauss}(SU_q(2)) = \mathbb{T}$. And combining other known results we get the following statement.

Proposition

Let G be a compact semisimple simply connected Lie group of rank k, let $q \in (0,1)$, and let $\mathbb{G}_q$ denote the Korogodsky-Soibelman deformation. Then

$$\mathrm{Gauss}(\mathbb{G}_q) = \mathbb{T}^k.$$

Dual case

Let Γ be a discrete group, and write $\mathrm{Pol}(\hat{\Gamma}) = \mathbb{C}[\Gamma]$. The finiteness/connectedness remarks imply that $\hat{\Gamma}$ cannot be Gaussian, if Γ has torsion.

Theorem

Let Γ be a discrete group. Then $\hat{\Gamma}$ is Gaussian if and only if it is torsion free nilpotent of class 2 (i.e. it is torsion free and all the commutators are central).

Thus for example the dual of the discrete Heisenberg group $\mathbb{H}_3$ is Gaussian.

Theorem

Let Γ be a discrete group. Then $\mathrm{Gauss}(\widehat{\Gamma}) = \widehat{\Gamma/\sqrt{\gamma_3(\Gamma)}}$, where the discrete group on the right hand side is a maximal torsion free nilpotent of class 2 quotient of Γ (and the normal subgroup $\sqrt{\gamma_3(\Gamma)}$ can be described explicitly).

We have two different proofs of the last theorem; we can either use the results of Passi on group rings or work directly with iterated commutators. The original discovery came from quantum stochastic ideas, the Heisenberg group and Weyl operators.

Further examples

- $\text{Gauss}(O_2^+) = \mathbb{T}$; $\text{Gauss}(O_N^+)$ is not classical for $N \geq 4$ (it contains both $\widehat{\mathbb{H}_3}$ and $SO(N)$)
- $\text{Gauss}(O_N^*) = SO_N$
- $\text{Gauss}(U_2^+) \supset \langle U_2, \widehat{\mathbb{H}_3} \rangle$

Above we use the notion of the quantum subgroup generated by two other quantum subgroups – this can be defined in terms of intersecting the relevant Hopf ideals.

Back to connectedness

Recall the ideals of $\mathrm{Pol}(\mathbb{G})$:

$$K_n := \mathrm{Lin}\{b_1 \cdots b_n : b_i \in \mathrm{Ker}(\epsilon)\},$$

$$K_\infty := \bigcap_{n \in \mathbb{N}} K_n.$$

Definition

We say that $\mathbb{G}$ is strongly connected if $K_\infty = \{0\}$; it is totally strongly disconnected if $K_\infty = K_1$.

A non-trivial fact is that K_∞ is always a coideal; in other words, there is a quantum subgroup $\mathbb{H}$ of $\mathbb{G}$ ('a strongly connected component of $\mathbb{G}$') such that $\mathrm{Pol}(\mathbb{H}) = \mathrm{Pol}(\mathbb{G})/K_\infty$.

Strongly connected quantum groups

Definition

We say that $\mathbb{G}$ is strongly connected if $K_\infty = \{0\}$; it is totally strongly disconnected if $K_\infty = K_1$.

- if $\mathbb{G}$ is Gaussian, then it must be strongly connected (but the converse does not hold);
- if Γ is discrete, then $\hat{\Gamma}$ is strongly connected if and only if it is residually torsion free nilpotent;
- U_N^+ is strongly connected
- if $\mathrm{Pol}(\mathbb{G})$ is generated by projections, then $\mathbb{G}$ is totally strongly disconnected
- so non-trivial finite quantum groups and quantum permutation groups S_N^+ are not Gaussian

Strongly connectedness vs connectedness

Definition (Wang, Cirio+D'Andrea+Pinzari+Rossi)

We say that $\mathbb{G}$ is connected if it does not admit any finite quotients (i.e. $\mathrm{Pol}(\mathbb{G})$ admits no finite-dimensional Hopf *-subalgebras)

- strong connectedness implies connectedness;
- both notions coincide for classical compact groups;
- but in general the converse implication does not hold.

One can naturally define and study strongly connected/connected components of identity for compact quantum groups.

What else do we know about Gaussian states?

- they satisfy a Wick type formula
- informally speaking the corresponding quantum Lévy processes involve the quantum Brownian motions (processes generated by combinations of annihilation and creation operators)
- if a generating functional on $\mathrm{Pol}(\mathbb{G})$ factors via a quantum subgroup $\mathbb{H}$, so does the associated convolution semigroup of states (and its limit at infinity, if it exists)

Questions/perspectives

- Is there any concrete Hopf-algebraic description of the Gaussian part of a given compact quantum group? Can one introduce abstractly 'iterated commutators' which would do the job?
- is U_N^+ (equivalently, U_2^+) Gaussian? Is O_N^+ strongly connected (say for $N \geq 4$)?
- One can ask similar questions for locally compact quantum groups á la Kustermans-Vaes. This raises the following (analytic) point: does the Gaussian property for a generating functional γ depend on choosing a dense *-subalgebra in the domain of γ?
- can one characterise/describe properties of Gaussian states in the sense proposed here beyond the properties of the associated generating functionals?
- very little is known about infinite divisibility for states on quantum groups!

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